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Complete convergence for dependent random variables: A review

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Abstract: The concept complete convergence was introduced in 1947 by Hsu and Robbins who proved that the sequence of arithmetic means of i.i.d. random variables converges completely to the expected value of the variables provided their variance is finite. The complete convergence of dependent random variables has been investigated by several authors, for example, Amini M. and Bozorgnia A. (2003), Li, Y.X. and Zhang, L. (2004), Chen, P. et al. (2007), Amini, M. et al. (2007), Wu, Q.Y. (2010), Ko, M.H. (2011), Amini, M. et al. (2012), Wang, X. et al. (2012), Yang, W. et al. (2012), Sung, S.H. (2012), Shen, A.T. et al.(2013) Wang, X. et al. (2014), Amini, M. et al. (2015), Wang, X. et al. (2015), Amini, M. et al. (2016), Deng, X. et al. (2016), and Amini, M. et al. (2017). In general, the main tools to prove the complete convergence of some random variables are based on Borel-Cantelli lemma and the moment inequality or the exponential inequality. However, for some dependent sequences (such as weakly negative dependent (WND) and negative superadditive-dependent random (NSD) sequence), whether these inequalities hold was not known. In this talk, we review complete convergence as historically from i.i.d. sequences to dependent sequences. In particular, complete convergence for weighted sums of weakly negative dependent are provided and applied to empirical distribution, sample p-th quantile and random weighting estimate. Also, the complete convergence is established for weighted sums of negatively superadditive-dependent random variables. Moreover, under the condition of integrability and appropriate conditions on the array of weights, the conditional mean convergence and conditional almost sure convergence theorems for weighted sums of an array of random variables are obtained when the random variables are special kind of dependence. As, some applications, complete convergence for weighted sums, moving average processes and the complete consistency of LS estimators in the EV regression model with NSD errors is investigated.

Keywords Complete convergence, Dependent random variables, Weighted sums, Moving average processes, EV regression model.

1 Introduction

One of the dependence structure that has attracted the interest of probabilist and statisticians is negative superaditive, we are interested in concept of conditional dependence, namely the concept of conditional negative superaditive dependence. The concept of conditional negative superadditive dependence is an extension to the conditional case of the concept of negative superadditive dependence introduced by Hu (2000) which was based on the class of superadditive functions. In this work, we extend the notion of (r, h)-integrable of $\{X_{nk}\}$ with respect to constants weights $\{a_{nk}\}$ to the corresponding conditional notion in the more general setting of randomly weighted sum of random variables (i.e., to the case in which the weights are also random variables $\{A_{nk}\}$) when a sequence of conditioning σ-algebras $\{\mathcal{F}_n\}$. We then obtain some results on conditional convergence of these sums given the conditioning σ-algebras of events $\{\mathcal{F}_n\}$ that extend, in a substantial way, the main mean convergence theorems in Ordóñez et al. (2012).

All events and random variables are defined on the same probability space $(\Omega, \mathcal{A}, P)$. Throughout, $\mathcal{F}$ and $\mathcal{F}_n,\ n \geq 1$ be sub-σ-algebras of $\mathcal{A}$ and we denote by $E^{\mathcal{F}}(X)$ the conditional expectation of the random variable X relative to $\mathcal{F}$, and by $P^{\mathcal{F}}(A)$ the conditional probability of the event $A \in \mathcal{A}$ relative to $\mathcal{F}$. Roussas (2008) provides a detailed proof of an integral representation of the covariance of two random variables, a brief proof of which is available in Lehmann (1966). The first propositions come from Roussas (2008).

Proposition 1.1. *Let random variables X and Y be have finite second moments. Then their conditional covariance, given $\mathcal{F}$, is defined by*

$$Cov^{\mathcal{F}}(X, Y) = E^{\mathcal{F}}[(X - E^{\mathcal{F}}X)(Y - E^{\mathcal{F}}Y)].$$

By applying a conditional version of the Fubini theorem, Roussas (2008) obtains the following integral representation for the conditional covariance of two random variables:

Proposition 1.2. *let X and Y be random variables with* $EX^2 < \infty$*. Then*

$$Cov^{\mathcal{F}}(X,Y) = \int_{R^2} H^{\mathcal{F}}(x,y)dx\ dy \quad a.s.,$$

where $H^{\mathcal{F}}(x,y) = P^{\mathcal{F}}[X \leq x, Y \leq y] - P^{\mathcal{F}}[X \leq x]P^{\mathcal{F}}[Y \leq y]$.

Proposition 1.3. *If the integrable random variables* X *and* Y *are* $\mathcal{F}$*-independent, then*

$$E^{\mathcal{F}}(XY) = E^{\mathcal{F}}(X)E^{\mathcal{F}}(Y) \quad a.s.,$$

and similarly for any finite number of random variables, i.e.

$$E^{\mathcal{F}}(\Pi_{i=1}^{n} X_i) = \Pi_{i=1}^{n} E^{\mathcal{F}}(X_i).$$

We now present the basic definitions and results concerning conditional negative superadditive dependence.

Definition 1.4. *(Ordóñez et al. 2012). Random variables X and Y are said to be conditionally negative quadrant dependent relative to a* σ*-algebra* $\mathcal{F}$ *(*$\mathcal{F}$*-CNQD) if*

$$P^{\mathcal{F}}[X \leq x, Y \leq y] \leq P^{\mathcal{F}}[X \leq x]P^{\mathcal{F}}[Y \leq y] \quad a.s.\ for\ all\ x,\ y \in R.$$

A sequence of random variables $\{X_n, \geq 1\}$ is said to be pairwise conditionally negative quadrant dependent relative to a σ-algebra $\mathcal{F}$ if every pair of random variables in the sequence is $\mathcal{F}-CNQD$.

Definition 1.5. *(Kemperman, 1997). The function* $\phi : \mathcal{R}^m \to \mathcal{R}$ *is called superadditive if* $\phi(\boldsymbol{x} \vee \boldsymbol{y}) + \phi(\boldsymbol{x} \wedge \boldsymbol{y}) \geq \phi(\boldsymbol{x}) + \phi(\boldsymbol{y})$ *for all* $\boldsymbol{x}, \boldsymbol{y} \in \mathcal{R}^m$*, where* $\vee$ *and* $\wedge$ *are for componentwise maximum, respectively.*

Lemma 1.6. *(Kemperman, 1997). If* ϕ *has continuous second partial derivatives, then superadditivity of* ϕ *is equivalent to* $\partial^2\phi/\partial x_i \partial y_j \geq 0,\ 1 \leq i \neq j \leq m$.

Remark 1.7. *By Lemma 1.6, when* $\{x_i\}$ *are non-negative real numbers, for all* $n \geq 2$ *and non-negative integer* s*, the function* $\phi(x_1, x_2, \cdots, x_n) = (\Pi_{k=1}^{n} x_k)^s$ *is a superadditive function.*

Now we give a new definition that it's relationship with the definition 1.4 will be mentioned in Proposition 2.2.

Definition 1.8. *Random variables X and Y are said to be conditionally negative superadditive dependent relative to a σ-algebra $\mathcal{F}$ ($\mathcal{F}$-CNSD) if*

$$E^{\mathcal{F}}(\phi(X,Y)) \leq E^{\mathcal{F}}(\phi(X^*,Y^*)),$$

where X^*, Y^* are independent with $X =^d X^*$, $Y =^d Y^*$ and ϕ is a superadditive function that the expectations exist.

A sequence of random variables $\{X_n, \geq 1\}$ is said to be pairwise conditionally negative supperadditive dependent relative to a σ-algebra $\mathcal{F}$ if every pair of random variables in the sequence is $\mathcal{F} - CNSD$.

Note that if $\mathcal{F} = \{\phi, \Omega\}$, then a sequence of pairwise $\mathcal{F}$-CNSD random variables is precisely a sequence of random variables which are negative superadditive dependent (NSD) in the unconditional case.

In the following, $\{u_n, n \geq 1\}$ and $\{v_n, n \geq 1\}$ will be two sequences of integers (not necessary positive or finite) such that $v_n \geq u_n$ for all $n \geq 1$ and $v_n - u_n \to \infty$ as $n \to \infty$. Moreover, $\{h(n), n \geq 1\}$ will be a sequence of positive constants with $h(n) \uparrow \infty$ *as* $n \to \infty$.

The notion of uniform integrability plays the central role in establishing L^r convergence and weak laws of large numbers. The classical notion of uniform integrability of a sequence $\{X_n, n \geq 1\}$ of integrable random variables is defined through the condition

$$\lim_{a\to\infty} \sup_{n\geq 1} E|X_n| I(|X_n| > a) = 0.$$

The concept of the uniform integrability has been generalized and extended in several directions for details see Cao (2013). We now introduce a new concept of integrability as follow:

Definition 1.9. *Let $r > 0$, $\{X_{nk}, u_n \leq k \leq v_n, n \geq 1\}$ and $\{A_{nk}, u_n \leq k \leq v_n, n \geq 1\}$ be two arrays of random variables. The array $\{X_{nk}\}$ is said to be conditionally residually (r,h)- integrable relative to $\mathcal{F}_n$ ($\mathcal{F}_n$-CR-(r,h)-integrable), concerning the array $\{A_{nk}\}$ if the following conditions hold:*

(a) $\sup_{n\geq 1} \sum_{k=u_n}^{v_n} |A_{nk}|^r E^{\mathcal{F}_n} |X_{nk}|^r < \infty$ *a.s.,*

(b) $\lim_{n\to\infty} \sum_{k=u_n}^{v_n} |A_{nk}|^r E^{\mathcal{F}_n}(|X_{nk}| - h(n))^r I[|X_{nk}| > h(n)] = 0 \quad a.s.$

It's easily seen that If $A_{nk} \equiv a_{nk}$ are constants, and $\mathcal{F}_n = \{\phi, \Omega\}$ for all $n \in N$, we have the concept of residual (r,h)-integrability concerning the array of constants $\{a_{nk}\}$.

Remark 1.10. *The $\mathcal{F}_n$-CR-(r,h)-integrable concerting the arrays $\{A_{nk}\}$ is weaker than the $\mathcal{F}_n$-(1,h)-integrable concerting the arrays $\{A_{nk}\}$ who defined by Ordóñez et al (2012).*

2 Conditional mean convergence for randomly weighted sums

In order to prove the main results in this section and next sections, we first present some immediate consequences.

Lemma 2.1. *Let (X, Y) be a random vector with joint conditionally distribution function $H^{\mathcal{F}}$ and marginal conditionally distribution functions $F^{\mathcal{F}}$ and $G^{\mathcal{F}}$ and let (X^*, Y^*) be a random vector with joint conditionally distribution function $H^{*\mathcal{F}}(x, y) = F^{\mathcal{F}}(x)G^{\mathcal{F}}(y)$. Suppose that ϕ is a superadditive function from $\mathcal{R}^2$ to $\mathcal{R}$ such that $E^{\mathcal{F}}\phi(X, Y)$ and $E^{\mathcal{F}}\phi(X^*, Y^*)$ exist. Then*

$$E^{\mathcal{F}}\phi(X, Y) - E^{\mathcal{F}}\phi(X^*, Y^*) = \int_{R^2} [H^{\mathcal{F}}(x, y) - F^{\mathcal{F}}(x)G^{\mathcal{F}}(y)]\ d\phi(x, y).$$

Where $F^{\mathcal{F}}(x) = P^{\mathcal{F}}(X \le x)$ and $G^{\mathcal{F}}(y) = P^{\mathcal{F}}(Y \le y)$.

Proof: *The proof is inspired by Theorem 2.1 of Molina (1992). Let (X_1, Y_1) and (X_2, Y_2) be independent and identically distributed with joint conditionally distribution function $H^{\mathcal{F}}$ and marginal conditionally distribution functions $F^{\mathcal{F}}$ and $G^{\mathcal{F}}$ that (X_1, Y_2) and (X_2, Y_1) have the joint conditionally function $F^{\mathcal{F}}(x).G^{\mathcal{F}}(y)$. It can be checked that*

$$\begin{aligned} &\phi(X_1, Y_1) - \phi(X_1, Y_2) - \phi(X_2, Y_1) + \phi(X_2, Y_2) \\ = &\int_{R^2} [I(X_1 > t) - I(X_2 > t)][I(Y_1 > s) - I(Y_2 > s)]\ d\phi(t, s). \end{aligned}$$

Since (X_1, Y_2) *and* (X_1, Y_2) are independent and identically distributed , to get

$$\begin{aligned} &E^{\mathcal{F}}[\phi(X_1, Y_1) - \phi(X_1, Y_2) - \phi(X_2, Y_1) + \phi(X_2, Y_2)] \\ &\qquad = 2[E^{\mathcal{F}}\phi(X_1, Y_1) - E^{\mathcal{F}}\phi(X_1, Y_2)] \\ &\qquad = 2[E^{\mathcal{F}}\phi(X, Y) - E^{\mathcal{F}}\phi(X^*, Y^*)] \end{aligned}$$

By Theorem 4.1 in Roussas (2008) (A conditional version of the Fubini Theorem), we have

$$\begin{aligned} & E^{\mathcal{F}}\int_{R^2}[I(X_1>t)-I(X_2>t)][I(Y_1>s)-I(Y_2>s)]\,d\phi(t,s) \\ = & \int_{R^2}E^{\mathcal{F}}[I(X_1>t)-I(X_2>t)][I(Y_1>s)-I(Y_2>s)]\,d\phi(t,s) \\ = & \ 2\int_{R^2}[P^{\mathcal{F}}(X_1>t,Y_1>s)-P^{\mathcal{F}}(X_1>t)P^{\mathcal{F}}(Y_2>s)]\,d\phi(t,s) \\ = & \ 2\int_{R^2}[P^{\mathcal{F}}(X_1\le t,Y_1\le s)-P^{\mathcal{F}}(X_1\le t)P^{\mathcal{F}}(Y_2\le s)]\,d\phi(t,s). \end{aligned}$$

This completes the proof.

Proposition 2.2. *For random variable vector* (X,Y)*,*

$$\mathcal{F}-CNSD(X,Y)\Leftrightarrow\mathcal{F}-CNQD(X,Y).$$

Prooof: Since $\phi(x,y)=I(x\le t,y\le s)$ for fixed real value t and s, is a superadditive function, hence, $\mathcal{F}$-CNSD$\Rightarrow$ $\mathcal{F}$-CNQD. The inverse valids by Lemma 2.1 and Definition 1.4.

Remark 2.3. *It's obvious from Proposition 1.2 that if* $\mathcal{F}-CNQD(X,Y)$ *then* $Cov^{\mathcal{F}}(X,Y)\le 0$*.*

Proposition 2.4. *If* (X_1,X_2) *is* $\mathcal{F}$*-CNSD and* g_1,g_2 *are increasing functions, then* $(g_1(X_1),g_2(X_2))$ *is* $\mathcal{F}$*-CNSD.*

Now, we state the main result and prove that.

Theorem 2.5. *Suppose* $\{X_{nk},u_n\le k\le v_n,n\ge 1\}$ *be an array of row-wise pairwise* $\mathcal{F}_n$*-CNSD random variables. Let* $\{A_{nk},u_n\le k\le v_n,n\ge 1\}$ *be an array of non-negative random variables such that, for each* $n\in N$*, the sequence* $\{A_{nk},u_n\le k\le v_n\}$ *are* $\mathcal{F}_n$*-measurable. Assume that the following conditions hold:*

(i) $\{X_{nk},u_n\le k\le v_n,n\ge 1\}$ *is* $\mathcal{F}_n$*-CR-(r,h)-integrable concerning the array* $\{A_{nk}\}$ *with exponent* $0<r\le 1$*,*

(ii) $h(n)(\sup_{u_n\le k\le v_n}A_{nk})\to 0$ *a.s. as* $n\to\infty$*.*

Let $S_n=\sum_{k=u_n}^{v_n}A_{nk}(X_{nk}-E^{\mathcal{F}_n}X_{nk})$*,* $n\ge 1$*. Then* $E^{\mathcal{F}_n}|S_n|^r\to 0$ *a.s. as* $n\to\infty$*.*

Proof: Consider continuous truncation, for $u_n\le k\le v_n,n\ge 1$,

$$X'_{nk}=-h(n)I[X_{nk}\le -h(n)]+X_{nk}I[|X_{nk}|\le h(n)]+h(n)I[X_{nk}>h(n)],$$

Moreover, denote

$$S_{1n} = \sum_{k=u_n}^{v_n} A_{nk}(X_{nk} - X_{nk}^{'}), \quad S_{2n} = \sum_{k=u_n}^{v_n} A_{nk}(X_{nk}^{'} - E^{\mathcal{F}_n} X_{nk}^{'}), \quad S_{3n} = \sum_{k=u_n}^{v_n} A_{nk} E^{\mathcal{F}_n}(X_{nk}^{'} - X_{nk}).$$

we write

$$S_n = S_{1n} + S_{2n} + S_{3n}, \quad n \geq 1,$$

and we can estimate the conditional expectation of each of these terms separately.

3 Conditional almost sure convergence for randomly weighted sums

To obtain a conditional strong convergence result, we introduce the concept of strongly conditionally residually (r,h)- integrable relative to $\mathcal{F}_n$ with exponent r as follows.

Definition 3.1. *Let $r > 0$, $\{X_{nk}, u_n \leq k \leq v_n, n \geq 1\}$ and $\{A_{nk}, u_n \leq k \leq v_n, n \geq 1\}$ be two arrays of random variables.The array $\{X_{nk}\}$ is said to be conditionally strongly residually (r,h)-integrable relative to $\mathcal{F}_n$ ($\mathcal{F}_n$-CSR-(r,h)-integrable), concerning the array $\{A_{nk}\}$ if the following conditions hold:*

(a) $\sup_{n\geq 1} \sum_{k=u_n}^{v_n} |A_{nk}|^r E^{\mathcal{F}_n}|X_{nk}|^r < \infty$ *a.s.,*

(b) $\sum_{n=1}^{\infty} \sum_{k=u_n}^{v_n} |A_{nk}|^r E^{\mathcal{F}_n}(|X_{nk}| - h(n))^r I[|X_{nk}| > h(n)] < \infty.$ *a.s.*

Remark 3.2. *If $A_{nk} \equiv a_{nk}$ are constants, and $\mathcal{F}_n = \{\phi, \Omega\}$ for all $n \in N$, the preceding definition reduce to the following new concept of strongly residually (r,h)- integrable concerning the array of constants $\{a_{nk}\}$.*

Definition 3.3. *Let $r > 0$, $\{X_{nk}, u_n \leq k \leq v_n, n \geq 1\}$ and $\{a_{nk}, u_n \leq k \leq v_n, n \geq 1\}$ an array of constants. The $\{X_{nk}\}$ is said to be strongly residually (r,h)- integrable (SR-(r,h)-integrable, for short) concerning the array $\{a_{nk}\}$ if the following conditions hold:*

(a) $\sup_{n\geq 1} \sum_{k=u_n}^{v_n} |a_{nk}|^r E|X_{nk}|^r < \infty$ *a.s.,*

(b) $\sum_{n=1}^{\infty} \sum_{k=u_n}^{v_n} |a_{nk}|^r E(|X_{nk}| - h(n))^r I[|X_{nk}| > h(n)] < \infty.$ *a.s.*

We will now establish a strong version of Theorem 2.5 under the condition of $\mathcal{F}$-$CNSD$ integrability ($i.e.$, when $\mathcal{B}_n \equiv \mathcal{B}$, a sub-$\sigma$- algebra of $\mathcal{A}$, for all $n \geq 1$).

Theorem 3.4. *Suppose $\{X_{nk}, u_n \leq k \leq v_n, n \geq 1\}$ be an array of row-wise pairwise $\mathcal{F}$-$CNSD$ random variables. Let $\{A_{nk}, u_n \leq k \leq v_n, n \geq 1\}$ be an array of non-negative random variables such that, for each $n \in N$, the sequence $\{A_{nk}, u_n \leq k \leq v_n\}$ are $\mathcal{F}$-measurable. Assume that the following conditions hold:*

(i) $\{X_{nk}, u_n \leq k \leq v_n, n \geq 1\}$ is $\mathcal{F}$-CSR-(r,h)-integrable concerning the array $\{A_{nk}\}$ with exponent $0 < r < 1$,

(ii) $\sum_{n=1}^{\infty} \left(h^r(n)(\sup_{u_n \leq k \leq v_n} A_{nk})\right)^{1-r} < \infty$ a.s.

Then $S_n = \sum_{k=u_n}^{v_n} A_{nk}(X_{nk} - E^{\mathcal{F}} X_{nk}) \to 0$ a.s. as $n \to \infty$.

Proof: We use the same notations as those in Theorem 2.5 and, set $\mathcal{F}_n = \mathcal{F}$ for each $n \in N$. Then $S_n = S_{1n} + S_{2n} + S_{3n}$ for each $n \in N$ and, we estimate each of these terms separately.

In the following, using Theorem 3.1 we obtain a conditional mean convergence. We assume that $\{Y_i, -\infty < i < \infty\}$ is a doubly infinite sequence of identically distributed random variables with $E|Y_1| < \infty$. Let $\{a_i, -\infty < i < \infty\}$ be an absolutely summable sequence of real numbers and

$$X_k = \sum_{i=-\infty}^{\infty} a_i Y_{k+i}, \qquad n \geq 1$$

be the moving average process based on the sequence $\{Y_i, -\infty < i < \infty\}$.

Asymptotic behavior for the moving average process $\{X_k; k \geq 1\}$ have been studied by many authors such as Li et al. (1992), Sung (1999), Sadeghi and Bozorgnia (1994) and Amini et al. (2015).

We will now establish mean convergence under condition $\mathcal{F}$-CNSD (i.e. when $\mathcal{F}_n = \mathcal{F}$, a sub-$\sigma$-algebra of $\mathcal{A}$, for all $n \in \mathbb{N}$)

Theorem 3.5. *Let $\{X_n, n \geq 1\}$ be a moving average process based on a sequence $\{Y_i, -\infty < i < \infty\}$ of identically distributed pairwise $\mathcal{F}$-CNSD random variables. Let $S_n = n^{-1}\sum_{k=1}^{n}(X_k - E^{\mathcal{F}} X_k)$, $n \geq 1$.*

(i) If $E^{\mathcal{F}}|Y_1| < \infty$, then $E^{\mathcal{F}}|S_n| \to 0$ a.s. as $n \to \infty$.

(ii) If $\sum_{n=1}^{\infty} E^{\mathcal{F}}|Y_1|I[|Y_1| > h(n)] < \infty$, *then* $S_n \to 0$ *a.s. as* $n \to \infty$.

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